Lecture 24

Andrei Antonenko

April 4, 2003

1 On the row(column) expansion

This lecture we will give a nice example of application of a row expansion to computing the determinant of a large matrices.

Let A_n be the matrix with n rows and n columns of the following form.

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{pmatrix}$$

So it is a matrix with 2's on a diagonal and 1's strictly above and below it. Our goal is to compute the determinant of A_n as a function of n.

First we will use expansion by the first row. We will have:

$$\det A_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & 1 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix}$$

$$= 2 \det A_{n-1} - \begin{vmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & 1 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{vmatrix}$$

For the last matrix we will use expansion by the first column. So, the initial determinant will be equal to:

$$\det A_n = 2\det A_{n-1} - 1 \cdot \begin{vmatrix} 2 & 1 & \dots & 0 & 0 \\ 1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & \dots & 1 & 2 \end{vmatrix}$$

This last matrix is a $(n-2) \times (n-2)$-matrix of the same type, so we got the following equality:

$$\det A_n = 2\det A_{n-1} - \det A_{n-2}.$$

Now let's compute $\det A_1$ and $\det A_2$.

$$\det A_1 = \begin{vmatrix} 2 \end{vmatrix} = 2; \qquad \det A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3.$$

Now by this formula we can compute larger determinants, e.g.:

$$\begin{aligned} \det A_3 &= 2\det A_2 - \det A_1 = 2 \cdot 3 - 2 = 4; \\ \det A_4 &= 2\det A_3 - \det A_2 = 2 \cdot 4 - 3 = 5; \\ \det A_5 &= 2\det A_4 - \det A_3 = 2 \cdot 5 - 4 = 6. \end{aligned}$$

Looking at these determinant we're able to guess the general formula:

$$\det A_n = n + 1.$$

It is not proved yet, it is just our guess. But we can use mathematical induction to prove it:

Proof. **Base of induction.** For $n = 1$ $\det A_1 = 2 = 1 + 1$ — formula is true. For $n = 2$ $\det A_2 = 3 = 2 + 1$ — formula is true.

Step of induction. Let the formula be true for $\det A_k$, i.e. $\det A_k = k + 1$ and $\det A_{k-1}$, i.e. $\det A_{k-1} = k$. Now, by our recurrent relation:

$$\det A_{k+1} = 2\det A_k - \det A_{k-1} = 2(k+1) - k = k + 2 = (k+1) + 1,$$

so, our guess is true in general. □

In this example we just guessed the general formula, and it may be difficult to do it in general, when it is more complicated. There is a whole theory how to solve recurrent relations (which is not difficult, on the contrary, it is quite short and simple), but in this course we will not cover it.

2 Applications of determinants

So, now we know how to compute determinants. In this chapter we will give 2 application of determinants, which are not very interesting from the practical point of view, but which is quite important from the theoretical.

2.1 Cramer's rule

Let's consider the following system of equations:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n \end{cases}$$

Let A be the matrix of coefficients and b be the right-hand side column vector:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

By A_i we will denote the matrix obtained from A by replacing its i-th with the column b.

Let $\det A \neq 0$. Then we can solve the system using the following formulae:

$$x_i = \frac{\det A_i}{\det A} \quad \forall i = 1, \dots, n.$$

These formulae are called **Cramer's rule**.

Example 2.1. *Consider the following system:*

$$\begin{cases} 2x_1 & + & 3x_2 & = & 12 \\ 3x_1 & - & x_2 & = & 7 \end{cases}$$

Here we have:

$$\det A = \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 2(-1) - 3 \cdot 3 = -11;$$

$$\det A_1 = \begin{vmatrix} 12 & 3 \\ 7 & -1 \end{vmatrix} = 12(-1) - 3 \cdot 7 = -33;$$

$$\det A_2 = \begin{vmatrix} 2 & 12 \\ 3 & 7 \end{vmatrix} = 2 \cdot 7 - 12 \cdot 3 = -22.$$

So, by Cramer's rule we have:

$$x_1 = \frac{\det A_1}{\det A} = \frac{-33}{-11} = 3; \quad x_2 = \frac{\det A_2}{\det A} = \frac{-22}{-11} = 2.$$

If $\det A = 0$ we can not apply Cramer's rule, so we're not able to solve the system by this method. Actually, in this case the system has either no solution or infinitely many solutions.

We will prove Cramer's rule in the addendum to this lecture.

2.2 Formula for the inverse

Last lecture we saw that if the matrix is invertible, then its determinant is not equal to 0. So, there exists the following formula for the inverse of the matrix.

Let A be an invertible matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \qquad \det A \neq 0.$$

Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}.$$

where A_{ij} is a cofactor of (i,j)-th entry of A.

In the case when A is a 2×2-matrix this gives us the familiar formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3 Proofs

Proof of the Cramer's rule. First let's note that if we apply an elementary operation to the system of the equations, then the same elementary row operation is applied to the matrix A and A_i. So, their determinant are changed similarly, i.e if the determinant of A changes its sign, then the determinants of all A_i's change their signs, and if the determinant of A is multiplied by $c \neq 0$, then the determinants of all A_i's are multiplied by the same number c. So, the relations $\dfrac{\det A_i}{\det A}$ from the Cramer's rule do not change. Using elementary operations we can transform the system to the form

$$\begin{cases} x_1 & & & = k_1 \\ & x_2 & & = k_2 \\ \dots & \dots & \dots & \dots \\ & & x_n & = k_n \end{cases}$$

It has the solution $x_i = k_i$. So, it is sufficient to prove Cramer's rule gives the same answer. For it:

$$\det A = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1; \quad \det A_i = \begin{vmatrix} 1 & 0 & \dots & k_1 & \dots & 0 & 0 \\ 0 & 1 & \dots & k_2 & \dots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \dots & k_i & \dots & 0 & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \dots & k_{n-1} & \dots & 1 & 0 \\ 0 & 0 & \dots & k_n & \dots & 0 & 1 \end{vmatrix} = k_i$$

(in the last matrix the column of b_i's stays on the place of i-th column). So, for this special type of system by Cramer's rule we obtain $x_i = \dfrac{\det A_i}{\det A} = k_1$. $\square$

Proof of the formula for the inverse. Matrix A^{-1} is the solution for the matrix equation

$$AX = E \tag{1}$$

Let X_i denotes the i-th column of the matrix X. Then the equation (1) is equivalent to the following n matrix equations with unknown columns of X:

$$AX_j = I_j \qquad \forall j = 1, \dots, n,$$

where I_j is the j-th column of the identity matrix. Each of the matrix equations $AX_j = I_j$ can be written as the following system with unknown entries $x_{1j}, x_{2j}, \dots, x_{nj}$ of X_j. The matrix of coefficients of this system is the matrix A, and the right-hand side is I_j. Now by Cramer's rule

$$x_{ij} = \frac{1}{\det A} \begin{vmatrix} a_{11} & \dots & 0 & \dots & a_{1n} \\ \multicolumn{5}{c}{\dotfill} \\ a_{j1} & \dots & 1 & \dots & a_{1n} \\ \multicolumn{5}{c}{\dotfill} \\ a_{n1} & \dots & 0 & \dots & a_{nn} \end{vmatrix} = \frac{A_{ji}}{\det A}$$

$\square$